Solutions to Chapter 1 of Introduction to Analysis by Maxwell Rosenlicht

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Chapter 1

Notions from Set Theory

1.1 Discussion

This section is all about naive set theory. It's been a little while since I was last working through Enderton's book, and even though I did learn a ton from it, there are still many gaps in my understanding and intuition. Even when it comes to simple set theoretic proofs, I sort of forgot that proving equality only requires you to show that an arbitrary element belongs to both sets. In other words, for the first exercise in the next section, it's really only necessary to find some b such that both $X^c \cap Y^c \subset (X \cup Y)^c$ and $(X \cup Y)^c \subset X^c \cap Y^c$ to show that these two sides are equal. This seems counterintuitive, but it's just because b is arbitrary, and could be any element.

I ended up actually having a lot of fun with this section. Something about the flow of these proofs just appeals to me. It makes me feel like I'm sure my students do; the first introduction to an idea is difficult and a bit scary even. Once you get used to it, though, and it starts to feel natural, there's this kind of austere beauty to it all that's difficult to describe but easy to appreciate.

The last thing I want to mention here is that due to this having only taken a week to write up, there are likely issues with the text and solutions. If you notice any, let me know and I'll fix them. Some of them may even be glaring. I haven't really had much of a chance to look in detail into any of these problems.

1.2 Thoughts From the Text

Exercise 1. *Prove that if $X \subset S$, $Y \subset S$, then $X^c \cap Y^c = (X \cup Y)^c$.*

Proof. First, let there exist some $b \in X^c \cap Y^c$. This means that $b \in X^c$ and also $b \in Y^c$. We see that this implies both $b \in S$ but $b \notin x$ and also $b \in S$ but $b \notin Y$. In other words, b cannot be in either set X or Y, but must be in the surrounding set S. So we see that $b \in S$ but $b \notin X \cup Y$, as stated. Therefore, $b \in (X \cup Y)^c$, and so $X^c \cap Y^c \subset (X \cup Y)^c$.

Conversely, suppose $b \in (X \cup Y)^c$. Then, $b \in S$ but $b \notin X \cup Y$, which implies that $b \in S$ but $b \notin X$ and $b \notin Y$. Since $b \in S$ and $b \notin X$, we have that $b \in X^c$. Likewise, since $b \in S$ and $b \notin Y$, we get $b \in Y^c$. Since we know that both cases must be true, we obtain that $b \in X^c \cap Y^c$, and so $(X \cup Y)^c \subset X^c \cap Y^c$. Since we have that each side is a subset of the other, we get equality, so that $X^c \cap Y^c = (X \cup Y)^c$. □

Exercise 2. *Prove that if I and S are sets and if for each $i \in I$ we have $X_i \subset S$, then*

$$\left(\bigcap_{i\in I} X_i\right)^c = \bigcup_{i\in I}(X_i)^c.$$

Proof. Let there exist some $a \in \left(\bigcap_{i\in I} X_i\right)^c$. This implies that even though $a \in S$, we have that $a \notin \bigcap_{i\in I} X_i$. Then, we have that it must be the case for some $k \in I$ that $a \in (X_k)^c$. Thus, we can state that $a \in \bigcup_{i\in I}(X_i)^c$. This means that we have $\left(\bigcap_{i\in I} X_i\right)^c \subset \bigcup_{i\in I}(X_i)^c$.

Conversely, suppose $a \in \bigcup_{i\in I}(X_i)^c$. This means that $a \in S$ and $a \in \bigcup_{i\in I}(X_i)^c$. By extension, there must exist some $k \in I$ for which $a \in (X_k)^c$. Thus, we have for at least this one index that $a \in \left(\bigcap_{i\in I} X_i\right)^c$, and so $\left(\bigcap_{i\in I} X_i\right)^c \subset \left(\bigcap_{i\in I} X_i\right)^c$. Thus, since we have that a is a member of both subsets, we obtain the equality we were looking for and state that $\left(\bigcap_{i\in I} X_i\right)^c = \bigcup_{i\in I}(X_i)^c$. □

1.3 Exercises

Exercise 3. *Let $\mathbb{R}$ be the set of real numbers and let the symbols $<$, $\leqslant$ have their conventional meanings.*

(a) Show that

$$\{x \in \mathbb{R} : 0 \leqslant x \leqslant 3\} \cap \{x \in \mathbb{R} : -1 < x < 1\} = \{x \in \mathbb{R} : 0 \leqslant x < 1\}.$$

(b) List the elements of

$$(\{2, 3, 4\} \cup \{x \in \mathbb{R} : x^2 - 4x + 3 = 0\}) \cap \{x \in \mathbb{R} : -1 \leqslant x < 3\}.$$

(c) Show that

$$(\{x \in \mathbb{R} : -2 \leqslant x \leqslant 0\} \cup \{x \in \mathbb{R} : 2 < x < 4\}) \cap \{x \in \mathbb{R} : 0 \leqslant x \leqslant 3\}$$

$$= \{x \in \mathbb{R} : 2 < x \leqslant 3\} \cup \{0\}.$$

Proof of (a). Let $a \in \{x \in \mathbb{R} : 0 \leqslant x \leqslant 3\} \cap \{x \in \mathbb{R} : -1 < x < 1\}$. By this we must have both that $a \in \{x \in \mathbb{R} : 0 \leqslant x \leqslant 3\}$ and $a \in \{x \in \mathbb{R} : -1 < x < 1\}$. Of course, by element inclusion we must have both that $a \geqslant 0$ and $a < 1$, so the intersection of these sets means precisely that $a \in \{x \in \mathbb{R} : 0 \leqslant x < 1\}$, which implies that $\{x \in \mathbb{R} : 0 \leqslant x < 1\} \subset \{x \in \mathbb{R} : 0 \leqslant x < 1\}$.

Now, suppose that $a \in \{x \in \mathbb{R} : 0 \leqslant x < 1\}$. Then, we can write that $a \in \{x \in \mathbb{R} : -1 < x < 1\}$ and also that $a \in \{x \in \mathbb{R} : 0 \leqslant x \leqslant 3\}$. Therefore, $a \in \{x \in \mathbb{R} : 0 \leqslant x \leqslant 3\} \cap \{x \in \mathbb{R} : -1 < x < 1\}$, and so $\{x \in \mathbb{R} : 0 \leqslant x \leqslant 3\} \cap \{x \in \mathbb{R} : -1 < x < 1\} \subset \{x \in \mathbb{R} : 0 \leqslant x \leqslant 1\}$. Because we have subsets from both directions, we can say that

$$\{x \in \mathbb{R} : 0 \leqslant x \leqslant 3\} \cap \{x \in \mathbb{R} : -1 < x < 1\} = \{x \in \mathbb{R} : 0 \leqslant x < 1\}.$$

□

Proof of (b). We must first find the elements of the second set. This is an easy task, as $x^2 - 4x + 3$ is factorable as $(x-1)(x-3)$, and so we can rewrite our sets as

$$(\{2, 3, 4\} \cup \{1, 3\}) \cap \{x \in \mathbb{R} : -1 \leqslant x < 3\}.$$

Now, see that this left union becomes the set $\{1, 2, 3, 4\}$, and so the intersection of this with the set $\{x \in \mathbb{R} : -1 \leqslant x < 3\}$ is just the set $\{1, 2\}$. □

Proof of (c). Let $a \in (\{x \in \mathbb{R} : -2 \leqslant x \leqslant 0\} \cup \{x \in \mathbb{R} : 2 < x < 4\}) \cap \{x \in \mathbb{R} : 0 \leqslant x \leqslant 3\}$. See that this implies that $a \in (\{x \in \mathbb{R} : -2 \leqslant x \leqslant 0\} \cup \{x \in \mathbb{R} : 2 < x < 4\})$, and so by union we have $a \in \{x \in \mathbb{R} : -2 \leqslant x \leqslant 0 \wedge 2 < x < 4\}$. Since a is in the intersection of this set and $\{x \in \mathbb{R} : 0 \leqslant x \leqslant 3\}$, we see that $a \in \{0\}$ and $a \in \{x \in \mathbb{R} : 2 < x \leqslant 3\}$. In other words, $a \in \{0\} \cup \{x \in \mathbb{R} : 2 < x \leqslant 3\}$, and so

$$(\{x \in \mathbb{R} : -2 \leqslant x \leqslant 0\} \cup \{x \in \mathbb{R} : 2 < x < 4\}) \cap \{x \in \mathbb{R} : 0 \leqslant x \leqslant 3\}$$

$$\subset \{x \in \mathbb{R} : 2 < x \leqslant 3\} \cup \{0\}$$

.

Taking the converse, suppose $a \in \{x \in \mathbb{R} : 2 < x \leqslant 3\} \cup \{0\}$. This means that either $a \in \{0\}$ or $a \in \{x \in \mathbb{R} : 2 < x \leqslant 3\}$. If $a \in \{0\}$ then we have that $a \in \{x \in \mathbb{R} : 0 \leqslant x \leqslant 3\}$, and so $a \in (\{x \in \mathbb{R} : -2 \leqslant x \leqslant 0\} \cup \{x \in \mathbb{R} : 2 < x < 4\}) \cap \{x \in \mathbb{R} : 0 \leqslant x \leqslant 3\}$. Supposing instead that $a \in \{x \in \mathbb{R} : 2 < x \leqslant 3\}$, then we also know that $a \in (\{x \in \mathbb{R} : -2 \leqslant x \leqslant 0\} \cup \{x \in \mathbb{R} : 2 < x < 4\})$. And because of this, we have $a \in (\{x \in \mathbb{R} : -2 \leqslant x \leqslant 0\} \cup \{x \in \mathbb{R} : 2 < x < 4\}) \cap \{x \in \mathbb{R} : 0 \leqslant x \leqslant 3\}$. Therefore,

$$\{x \in \mathbb{R} : 2 < x \leqslant 3\} \cup \{0\}$$

$$\subset (\{x \in \mathbb{R} : -2 \leqslant x \leqslant 0\} \cup \{x \in \mathbb{R} : 2 < x < 4\}) \cap \{x \in \mathbb{R} : 0 \leqslant x \leqslant 3\}$$

Since we have subsets on either side, we have the equality we're looking for. □

Exercise 4. *If A is a subset of the set S, show that*

(a) $(A^c)^c = A$

(b) $A \cup A = A \cap A = A \cup \varnothing = A$

(c) $A \cap \varnothing = \varnothing$

(d) $A \times \varnothing = \varnothing$.

Proof of (a). Let $a \in (A^c)^c$. Then, we must have that $a \notin A^c$, which implies that $a \in A$. This means that $(A^c)^c \subset A$. To prove the converse, take that $a \in A$. We see therefore that $a \notin A^c$, and by another application of the complement, we have $a \in (A^c)^c$. Therefore, $A \subset (A^c)^c$, and so $(A^c)^c = A$. □

Proof of (b). Take first that $a \in A$. Then, we have automatically by definition of union that $a \in A \cup \varnothing$, and so $A \subset A \cup \varnothing$. To show the other direction, suppose $a \in A \cup \varnothing$. Then, either $a \in A$ or $A \in \varnothing$. If $a \in \varnothing$, then it is vacuously true that $a \in A$ as well. Thus, we have that $A \cup \varnothing \subset A$, and so $A \cup \varnothing = A$.

Now, let's show that $A \cap A = A$. First, suppose $a \in A \cap A$. Then, it is obviously true that $a \in A$, by intersection, and so $A \cap A \subset A$. Then, let $a \in A$. Here, see that a is automatically in the intersection of A with itself, and so $A \subset A \cap A$. Therefore, $A = A \cap A$. Thus, we can say that $A \cap A = A \cup \varnothing = A$.

All that remains to see is that $A \cup A = A$. Let $a \in A \cup A$. Then, it must be that $a \in A$, by definition of union, and so $A \cup A \subset A$. Now allow that $a \in A$, and then see that it must be obviously true that $a \in A \cup A$ as well. Therefore, we see that $A \subset A \cup A$, and so $A = A \cup A$. Therefore, we have the final statement that $A \cup A = A \cap A = A \cup \varnothing = A$. □

Proof of (c). Let $a \in A \cap \varnothing$. This means that both $a \in A$ and $a \in \varnothing$. Since $\varnothing \subset A$, it must be that $a \in \varnothing$. Therefore, $A \cap \varnothing \subset \varnothing$. Now, suppose that $A \in \varnothing$. Since $\varnothing \subset A$, we have by necessity that $a \in A \cap \varnothing$. So, see that $\varnothing \subset A \cap \varnothing$. Since we have subsets from both sides, we see that $A \cap \varnothing = \varnothing$. □

Proof of (d). We can define that $A \times \varnothing = \{(a, b) : a \in A, b \in \varnothing\}$. Since there is no b such that $b \in \varnothing$, it is vacuously true that $A \times \varnothing \subset \varnothing$. Now, let $a \in \varnothing$. Since there are no elements within $\varnothing$, it is again vacuously true that $\varnothing \subset A \times \varnothing$. Thus, we have that $A \times \varnothing = \varnothing$. □

Exercise 5. *Let A, B, C be subsets of a set S. Prove the following statements and illustrate them with diagrams(not going to do that).*

(a) $A^c \cup B^c = (A \cap B)^c$.

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(c) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof of (a). First, allow that $a \in A^c \cup A^c$. This means that $a \in A^c$ or $a \in B^c$, or both. Let's suppose first that $a \in A^c$. Then, $a \notin A$, and so we have trivially by intersection that $a \notin A \cap B$. This implies, of course, that $a \in (A \cap B)^c$. In the other case of $a \in B^c$, we have that $a \notin B$. Thus, we get again that $a \notin A \cap B$, and so $a \in (A \cap B)^c$. In either case, we conclude that $A^c \cup B^c \subset (A \cap B)^c$.

Conversely, say that $a \in (A \cap B)^c$. This suggests that $a \notin A \cap B$. Here, notice that this really means $a \notin A$ or $a \notin B$, or both. Suppose first that $a \notin A$. Then, by definition of complement, we have that $a \in A^c$. It is trivial by use of the union operator that $a \in A^c \cup B^c$. Taking the other possibility, suppose $a \notin B$. This implies that $a \in B^c$ by the same reasoning as before, and thus $a \in A^c \cup B^c$. Since we get the same result either way, we have that $(A \cap B)^c \subset A^c \cup B^c$.

Because we have subsets from either direction, we can say that $A^c \cup B^c = (A \cap B)^c$, as required. □

Proof of (b). First suppose that there exists an element a such that $a \in A \cap (B \cup C)$. This means that we have both $a \in A$ and $a \in B \cup C$. So, we have the two possibilities that either $a \in A$ and $a \in B$ or that $a \in A$ and $a \in C$. Assume the first case. Since $a \in A$ and $a \in B$, we know that $a \in A \cap B$. It is then trivial by the union operator that $a \in (A \cap B) \cup (A \cap C)$. Now, assume the second case. Here we get that $a \in A$ and $a \in C$. By similar logic as before, we see that $a \in A \cap C$, and so it is obviously true that $a \in (A \cap B) \cup (A \cap C)$, by virtue of the union operator. Therefore, we have that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.

Now assume conversely that $a \in (A \cap B) \cup (A \cap C)$. If $a \in A \cap B$ then we have that $a \in A$ and $a \in B$, whereas if $a \in A \cap C$, we see that $a \in A$ and $a \in C$. In either case, $a \in A$. So, suppose we have the first case. Since $a \in B$, we know trivially that $a \in B \cup C$. Because it is also true that $a \in A$, we get that $a \in A \cap (B \cup C)$. Assuming the second case, we have that since $a \in C$, we know also that $a \in B \cup C$. Because it is also true that $a \in A$, we get the intersection that $a \in A \cap (B \cup C)$. Since either case leads to the same result, we can state that $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$. Note that each side is a subset of the other, and so we can make our statement that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, as needed. □

Proof of (c). We are to show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. First, suppose that $a \in A \cup (B \cap C)$. This means that either $a \in A$ or $A \in B \cap C$, or both. Assume the former, and see that since $a \in A$, we must also have that $a \in A \cup B$ and also that $a \in A \cup C$. This is simply by definition of the union operator. Since a is in both, we know that $a \in (A \cup B) \cap (A \cup C)$. Now, suppose the latter and say that $a \in B \cap C$. This implies that both $a \in B$ and $a \in C$. Since $a \in B$, we know by union that $a \in A \cup B$. Likewise, because $a \in C$, we have that $a \in A \cup C$. Therefore, since a is in both sets, we have $a \in (A \cup B) \cap (A \cup C)$. All this is to say that $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.

Conversely, suppose that $a \in (A \cup B) \cap (A \cup C)$. This implies that both $a \in A \cup B$ and $a \in A \cup C$. Note that in either case, we get $a \in A$. Therefore, we have two cases; that either $a \in A$ or that $a \in B$ and $a \in C$. There are other combinations as well, such as $a \in A$ and $a \in B$, but these are trivial, as element a being a member of A means it does not matter whether it is also a member of B or C. So, first suppose that $a \in A$. Almost trivially, by the union operator, we also have that $a \in A \cup (B \cap C)$. Now, supposing the second case, that $a \in B$ and $a \in C$, we see also that $a \in B \cap C$. Again by the union operator, we find that $a \in A \cup (B \cap C)$. Therefore, we conclude that

$(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$. We have subsets on either side, and so we can make the statement that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, as needed. $\square$

Exercise 6. *If A, B, C are sets, show that*

(a) $(A - B) \cap C = (A \cap C) - B$

(b) $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$

(c) $A - (B - C) = (A - B) \cup (A \cap B \cap C)$

(d) $(A - B) \times C = (A \times C) - (B \times C)$

Proof of (a). We are to show that $(A - B) \cap C = (A \cap C) - B$. First, suppose $a \in (A-B) \cap C$. This means that $a \in A - B$ and also $a \in C$. Since $a \in A - B$, we know that $a \in A$ but $a \notin B$. We have that $a \in A$ and $a \in C$, and so $a \in A \cap C$. Of course, because $a \notin B$, we see that $a \in (A \cap C) - B$. Thus, we have that $(A - B) \cap C \subset (A \cap C) - B$.

Now suppose that $a \in (A \cap C) - B$. We have that $a \in A$ and $a \in C$, but $a \notin B$. Since $a \in A$ and $a \notin B$, we see that $a \in (A - B)$. Of course, since we now have that $a \in (A - B)$ and $a \in C$, we see that $a \in (A - B) \cap C$. Therefore, $(A \cap C) - B \subset (A - B) \cap C$, and so $(A - B) \cap C = (A \cap C) - B$. $\square$

Proof of (b). We are to show that $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$. First, allow that $a \in (A \cup B) - (A \cap B)$. This means that $a \in (A \cup B)$ but $a \notin (A \cap B)$. This second term can be rewritten as $a \in (A \cap B)^c$, which we have from Exercise 5 that this is simply equal to $A^c \cup B^c$. Therefore, $a \in A^c \cup B^c$. So, we know that $a \in (A \cup B)$ and also $a \in A^c \cup B^c$. If $a \in A$ and $a \in B^c$, then we have that $a \in A$ and $a \notin B$, or rather that $a \in A - B$. Likewise, if $a \in B$ and $a \in A^c$, this implies that $a \in B$ and $a \notin A$, so therefore $a \in B - A$. Thus, we know that $a \in (A-B) \cup (B-A)$. This implies that $(A \cup B) - (A \cap B) \subset (A-B) \cup (B-A)$, as needed.

Proceeding from the other direction, suppose that $a \in (A - B) \cup (B - A)$. This implies that either $a \in (A - B)$ or $a \in (B - A)$. Let's assume first that $a \in (A - B)$. We have that because of this, $a \in A$ and also $a \notin B$. In other words, we have almost trivially by the union operator that $a \in A \cup B$. Likewise, since $a \notin B$, then we also know that $a \notin A \cap B$. If a is not in B, then it certainly cannot be in the intersection of sets A and B. Being that $a \in (A \cup B)$ and $a \notin (A \cap B)$, we have therefore that $a \in (A \cup B) - (A \cap B)$. We can conclude that $(A - B) \cup (B - A) \subset (A \cup B) - (A \cap B)$.

Now, we need only assume the alternative case, that $a \in (B - A)$. This implies that $a \in B$ but that $a \notin A$. Since $a \in B$, we know also that $a \in A \cup B$. Likewise, because $a \notin A$, we have that $a \notin A \cap B$. Here, by similar reasoning used in the previous paragraph, we see that $a \in (A \cup B) - (A \cap B)$, and so again we see that $(A - B) \cup (B - A) \subset (A \cup B) - (A \cap B)$. Since we have subsets on both sides, we conclude that $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$. $\square$

Proof of (c). We are to show that $A - (B - C) = (A - B) \cup (A \cap B \cap C)$. First, suppose that $a \in A - (B - C)$, which implies that $a \in A$ and $a \notin B - C$. We can rewrite this last term as $a \notin B \cap C^c$, which may be helpful for our comprehension. This can be further separated out into the statements $a \notin B$ and $a \notin C^c$. Of course, this last statement is equivalent to $a \in C$. Since we have that $a \in A$ and $a \notin B$, we see that $a \in (A - B)$. By the union operator, it is obvious to state therefore that $a \in (A - B) \cup (A \cap B \cap C)$. This implies that $A - (B - C) \subset (A - B) \cup (A \cap B \cap C)$.

Conversely, take that $a \in (A - B) \cup (A \cap B \cap C)$. This means that either $a \in (A - B)$ or $a \in (A \cap B \cap C)$, or both. Assume first that $a \in (A - B)$. Then, we see that $a \in A$ but $a \notin B$. Since $a \notin B$, it is obvious that $a \notin B \cap C^c$ as well, since if a is not in B, it certainly cannot be in the intersection of B and C^c. This is a bit of an odd phrasing, but the reason we did this is because it is given in the text that $B - C = B \cap C^c$, and so we know that as a result, $a \notin B - C$. Now, since $a \in A$ but $a \notin (B - C)$, we see that $a \in A - (B - C)$, as needed.

We need only to take the other case, that $a \in (A \cap B \cap C)$, and show that this statement holds as well. Decoupling this statement, we get that $a \in A$, $a \in B$, and $a \in C$. We can rewrite this last statement as $a \notin C^c$, so that therefore $a \notin B \cap C^c$ as well. If a is not in C^c, then it certainly cannot be in the intersection of this set and another, namely B. Of course, this is simply the statement $a \notin B - C$. Since we have, at last, that $a \in A$ but $a \notin B - C$, we see that $a \in A - (B - C)$, which allows us to finally say that $(A - B) \cup (A \cap B \cap C) \subset A - (B - C)$.

Because we have bidirectional subsets, we have that

$$A - (B - C) = (A - B) \cup (A \cap B \cap C).$$

$\square$

Proof of (d). We are to show that $(A - B) \times C = (A \times C) - (B \times C)$. First, let there be some ordered pair $(a, b) \in (A - B) \times C$, which can be rewritten as

$$(A - B) \times C = \{(a, b) | a \in (A - B) \ \ \& \ \ b \in C\}.$$

Here we see that the term $a \in (A - B)$ is just the same as $a \in A$ and $a \notin B$, and so this set can be further expanded as

$$(A - B) \times C = \{(a, b) | a \in A \ \ \& \ \ a \notin B \ \ \& \ \ b \in C\}.$$

Since $a \in A$ and $b \in C$, we know also that $(a, b) \in A \times C$. Likewise, since $a \notin B$ and $b \in C$, we have $(a, b) \notin B \times C$. We can therefore conclude that $(a, b) \in (A \times C) - (B \times C)$, as needed. This shows that $(A - B) \times C \subset (A \times C) - (B \times C)$.

Now suppose that $(a, b) \in (A \times C) - (B \times C)$. This implies that $(a, b) \in A \times C$ but that $(a, b) \notin B \times C$. By definition of the cartesian product, we therefore know that $a \in A$ and $b \in C$. Since we know that $b \in C$, it must be that $a \notin B$. We now have that $a \in A$ but $a \notin B$, and so we know that $a \in A - B$. Likewise, since $b \in C$, we can see clearly that $(a, b) \in (A - B) \times C$, as needed. Thus, $(A \times C) - (B \times C) \subset (A - B) \times C$, and so since we have subsets in both directions, we can say that $(A - B) \times C = (A \times C) - (B \times C)$. $\square$

Exercise 7. *Let I be a nonempty set and for each $i \in I$, let X_i be a set. Prove that*

(a) for any set B we have

$$B \cap \bigcup_{i \in I} X_i = \bigcup_{i \in I} (B \cap X_i).$$

(b) if each X_i is a subset of a given set S, then

$$(\bigcup_{i \in I} X_i)^c = \bigcap_{i \in I} (X_i)^c.$$

Proof of (a). We are to show that for any set B we have

$$B \cap \bigcup_{i \in I} X_i = \bigcup_{i \in I} (B \cap X_i).$$

To begin with, let there exist some a such that $a \in B \cap \bigcup_{i \in I} X_i$. This implies that $a \in B$ and $a \in \bigcup_{i \in I} X_i$. By the definition of union, we know there must exist some index $k \in I$ such that $a \in X_k$. Thus, since $a \in B$ and $a \in X_k$, we know that $a \in B \cap X_k$. Almost trivially, we see therefore because since this statement is true for one index k, we can state that $a \in \bigcup_{i \in I} (B \cap X_i)$. So, we see clearly that $B \cap \bigcup_{i \in I} X_i \subset \bigcup_{i \in I} (B \cap X_i)$.

Proceeding from the other direction, suppose $a \in \bigcup_{i \in I} (B \cap X_i)$. Then, we must have for some index $k \in I$ that $a \in B \cap X_k$, by definition of union. Here we see that $a \in B$ and $a \in X_k$, and so since $a \in X_k$ for at least one index k, we have that $a \in \bigcup_{i \in I} X_i$. Finally, since a must be in both sets, we have that $a \in B \cap \bigcup_{i \in I} X_i$. Thus, we can conclude that $\bigcup_{i \in I} (B \cap X_i) \subset B \cap \bigcup_{i \in I} X_i$.

Because we have bidirectional subsets, we conclude that

$$B \cap \bigcup_{i \in I} X_i = \bigcup_{i \in I} (B \cap X_i).$$

□

Proof of (b). We are to prove that if each X_i is a subset of a given set S, then

$$(\bigcup_{i \in I} X_i)^c = \bigcap_{i \in I} (X_i)^c.$$

Let $a \in (\bigcup_{i \in I} X_i)^c$ to begin with. Thus, we see by definition of set complement that $a \notin \bigcup_{i \in I} X_i$. Take some index $k \in I$ and see that since $a \notin \bigcup_{i \in I} X_i$, it must be the case that $a \notin X_k$ as well, for at least this one index k. Since each $X_i \subset S$, we can suppose that $a \in (X_k)^c$, such that a is somewhere else in S. Finally, see that since a exists in at least this one subset of S, it is readily apparent that $a \in \bigcap_{i \in I} (X_i)^c$. This, of course, implies that $(\bigcup_{i \in I} X_i)^c \subset \bigcap_{i \in I} (X_i)^c$.

Conversely, suppose $a \in \bigcap_{i \in I} (X_i)^c$, instead. Then, there is at least one index $k \in I$ such that $a \in (X_k)^c$, by definition of intersection. Of course, this implies that $a \notin X_k$, and so we can conclude almost trivially that for at least this one index k, $a \notin \bigcup_{i\in I} X_i$. Then, this can be simply rewritten in its complement form as $a \in (\bigcup_{i\in I} X_i)^c$, as needed. This shows that $\bigcap_{i\in I}(X_i)^c \subset \bigcup_{i\in I} X_i$ and completes the proof, giving the equality

$$B \cap \bigcup_{i\in I} X_i = \bigcup_{i\in I}(B \cap X_i).$$

□

Exercise 8. *Prove that if $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$ are functions, then*

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Proof. We can accomplish this by seeing that each chain of composition produces the same element. First of all, see that because $f : X \to Y$ and $g : Y \to Z$, $g \circ f : X \to Z$. Now, Composing this with $h : Z \to W$, we see that $h \circ (g \circ f) : X \to W$. Let's compare this with the righthand side. First composing h with g, we get $h \circ g : Y \to W$. Then, composing this again with f, obtain $(h \circ g) \circ f : X \to W$. Since both compositions have the same domain and range, it makes sense to suggest that they are equal. To test this, we say that there exists some $x \in X$ and then see that

$$h \circ (g \circ f)(x) = h(g \circ f)(x) = h(g(f(x))) = h \circ g(f(x)) = (h \circ g) \circ f(x).$$

□

Exercise 9. *Let $f : X \to Y$ be a function, let A and B be subsets of X, and let C and D be subsets of Y. Prove that*

(a) $f(A \cup B) = f(A) \cup f(B)$

(b) $f(A \cap B) \subset f(A) \cap f(B)$

(c) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$

(d) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

(e) $A \subset f^{-1}(f(A))$

(f) $f(f^{-1}(C)) \subset C$

Proof of (a). First, let there exist some element a such that $a \in f(A \cup B)$. Then, we must have $a = f(b)$ for $b \in A \cup B$, implying either $b \in A$ or $b \in B$. This means that $a \in f(A)$ or $a \in f(B)$, which can be rewritten as $a \in f(A) \cup f(B)$. Therefore, $f(A \cup B) \subset f(A) \cup f(B)$.

Conversely, suppose $a \in f(A) \cup f(B)$. Then, we have that either $a \in f(A)$ or $a \in f(B)$. This suggests that there exists some $b \in A$ or $b \in B$ such that $a = f(b)$. Therefore, $a \in f(A \cup B)$, and so $f(A) \cup f(B) \subset f(A \cup B)$, implying equality. □

Proof of (b). Assume there exists some $a \in f(A \cap B)$. Then, there must exist some $b \in A \cap B$ such that $a = f(b)$. This implies that both $b \in A$ and $b \in B$, and so we have also that $a \in f(A)$ and $a \in f(B)$. Therefore, we see that $a \in f(A) \cap f(B)$, and so $f(A \cap B) \subset f(A) \cap f(B)$. □

Proof of (c). Suppose that $a \in f^{-1}(C \cup D)$. Then, we must have by definition of function inverse that $f(a) \in C \cup D$. This implies that either $f(a) \in C$ or $f(a) \in D$. Therefore, $a \in f^{-1}(C)$ or $a \in f^{-1}(D)$, implying that $a \in f^{-1}(C) \cup f^{-1}(D)$. This, of course, means that $f^{-1}(C \cup D) \subset f^{-1}(C) \cup f^{-1}(D)$.

Conversely, suppose $a \in f^{-1}(C) \cup f^{-1}(D)$. This means that either $a \in f^{-1}(C)$ or $a \in f^{-1}(D)$. By definition of function inverse, we have that $f(a) \in C$ or $f(a) \in D$. Clearly, this implies through the union operator that $f(a) \in C \cup D$, and so $a \in f^{-1}(C \cup D)$. Therefore, $f^{-1}(C) \cup f^{-1}(D) \subset f^{-1}(C \cup D)$. Since we have bidirectional subsets, we conclude that the subsets are equal. □

Proof of (d). We are to prove that $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$. First, let that $a \in f^{-1}(C \cap D)$. This implies by way of function inverses that $f(a) \in C \cap D$, and so $f(a) \in C$ and $f(a) \in D$. We see here that $a \in f^{-1}(C)$ and $a \in f^{-1}(D)$, and so $a \in f^{-1}(C) \cap f^{-1}(D)$ as well. Thus, we have that $f^{-1}(C \cap D) \subset f^{-1}(C) \cap f^{-1}(D)$.

Conversely, suppose $a \in f^{-1}(C) \cap f^{-1}(D)$. This implies that $a \in f^{-1}(C)$ and $a \in f^{-1}(D)$, and so therefore that $f(a) \in C$ and $f(a) \in D$. Thus, $f(a) \in C \cap D$, giving also that $a \in f^{-1}(C \cap D)$. Since this implies that $f^{-1}(C) \cap f^{-1}(D) \subset f^{-1}(C \cap D)$, we have equality of these two subsets. □

Proof of (e). We will show that $A \subset f^{-1}(f(A))$. Suppose that $a \in A$. This implies, of course, that $f(a) \in f(A)$. Now, since $f(A) \subset Y$, we know that $a \in f^{-1}(f(A))$, as needed. Therefore, we get $A \subset f^{-1}(f(A))$. □

Proof of (f). We aim to prove that $f(f^{-1}(C)) \subset C$. Let $a \in f(f^{-1}(C))$ to begin with, and let there be some element $b \in f^{-1}(C)$ such that $a = f(b)$. Then, we see equivalently that $f(b) \in C$. However, since $a = f(b)$, we have that $a \in C$, and so $f(f^{-1}(C)) \subset C$. □

Exercise 10. *Under the assumptions of Exercise 9, prove that f is injective if and only if the sign $\subset$ in (e) can be replaced by $=$ for all $A \subset X$, and f is surjective if and only if the sign $\subset$ in (f) can be replaced by $=$ for all $C \subset Y$.*

Proof. First, let us suppose that f is injective. We are to show that therefore, the $\subset$ symbol can be replaced by equality. Thus, because f is injective, we know that $f(x_2) = f(x_1)$ implies that $x_1 = x_2$. Now, suppose that $x_1 \in f^{-1}(f(A))$ and $f(x_1) \in f(A)$. Of course, this implies that there exists some element $x_2 \in A$ such that $f(x_2) = f(x_1)$. Because of this we can see that $x_2 = x_1$ by virtue of our function being one-to-one. Since $x_2 \in A$, we therefore know that $x_1 \in A$ as well, and that therefore $f^{-1}(f(A)) \subset A$. Since we had proven in part (e) the first direction of this equality, we can say that $A = f^{-1}(f(A))$ when the function f is one-to-one.

Let us now show that if the subset symbol is replaced by equality, the function f is one-to-one. We'll prove this using the contrapositive, thereby supposing that f is not one-to-one and showing that therefore we cannot replace the $\subset$ symbol by equality. Because f is not one-to-one, there exist some x_1, x_2 such that $x_1 \neq x_2$ but that $f(x_1) = f(x_2)$. Let $x_1 \in f^{-1}(f(A))$ and $x_2 \in A$, since these are both subsets of X. Then, since $x_1 \neq x_2$, we have that $A \neq f^{-1}(f(A))$, as needed.

We have completed the proof that equality implies injectivity, and now seek to show that in part (f) replacing the $\subset$ symbol with equality implies surjectivity, and vice-versa. First, let us suppose that f is surjective and show that this means $f(f^{-1}(C)) = C$. Note that because of part (f), we only need to demonstrate that $C \subset f(f^{-1}(C))$. Since f is surjective, this means that for every $x_1 \in C$ there exists some $x_2 \in f^{-1}(C)$ such that $f(x_2) = x_1$. This shows us that $x_2 = f^{-1}(x_1)$. However, since $x_1 \in C$, we see that $f^{-1}(x_1) \in f^{-1}(C)$, and so $x_2 \in f^{-1}(C)$. Of course, this can be rewritten by taking functions as $f(x_2) \in f(f^{-1}(C))$, and then since $x_1 = f(x_2)$, we get that $x_1 \in f(f^{-1}(C))$, as needed. Therefore, we can state equality

Let's now prove the other direction, that if the $\subset$ symbol is replaced with equality, the function f is surjective. Here, just as with the other section of this proof, we utilize the contrapositive. Thus, we are trying to show that if f is not surjective, the $\subset$ symbol cannot be replaced with equality, and thus $C \not\subset f(f^{-1}(C))$. Since f is not surjective, there is some $y \in C$ such that for all $x \in f^{-1}(C)$, $f(x) \neq y$. Of course, since we also have that $f(x) \in f(f^{-1}(C))$, by taking functions on both sides, and that $f(x) \neq y$, we see that $y \notin f(f^{-1}(C))$. So, we conclude that $C \not\subset f(f^{-1}(C))$, and therefore have completed the sufficient part of the biconditional. $\square$

Exercise 11. *How many subsets are there of the set $\{1, 2, 3, ..., n\}$? How many maps of this set into itself? How many maps of this set onto itself?*

Answer 1. *There are 2^n subsets of this set. This is known as the power set. To prove this, we'll use weak induction. For the base case, let $n = 1$. Then, there should only be 2^1 subsets, which we see are the subsets $\{\}$ and $\{1\}$. For the inductive step, assume that it works for 2^n. We will show that it also holds for 2^{n+1}. Suppose we have the sets $A = \{1, 2, 3, ..., n\}$ and $B = \{1, 2, 3, ..., n, n+1\}$. Then, see that $B = A \cup \{n + 1\}$. By the inductive hypothesis, we see that there are 2^n subsets of A. Now, because $B \subset A$, we know that there are 2^n subsets of A without $\{n + 1\}$ included, and another 2^n subsets of A with $\{n + 1\}$ as an element. Therefore, there are $2^n + 2^n = 2 * 2^n = 2^{n+1}$ subsets of A, as needed.*

Having an into mapping means that for each element in the set there is exactly one corresponding element. Here, see that there are n possible ways to map element 1 to the remaining elements, $n-1$ possible ways to map element 2 to those remaining, $n-2$ ways of mapping element 3 to remaining elements, and so on. Therefore, there are $n!$ different possibilities for an injective mapping.

Notice with surjective maps that it also has to be injective, as both sets have the same cardinality. Thus, we conclude likewise that there are $n!$ different options.

Exercise 12.

(a) *How many functions are there from a nonempty set S into $\varnothing$?*

(b) *How many functions are there from $\varnothing$ into an arbitrary set S?*

(c) *Show that the notation $\{X_i\}_{i \in I}$ implicitly involves the notion of function.*

Answer 2. *(a) Since a function from S to $\varnothing$ can be defined as a subset of $S \times \varnothing$, and clearly this subset is empty, we can conclude that there are no such functions. Recall that a function would take any $s \in S$ to one and only one $b \in \varnothing$, but of course there is no such b.*

(b) By comparison, this statement is vacuous for one function. If we take $\varnothing \times S$, we see that there is indeed some $s \in S$ for every $b \in \varnothing$. This is normally known as the empty function.

(c) Here we just note that $\{X_i\}_{i \in I}$ is a set given by some index set I and another set X. We can think here of the function $f : I \to X$, where $f(i) = \{X_i\}_{i \in I}$ for some $i \in I$. In this way, elements $i \in I$ under the function return elements $X_i \in X$.