Derivative of the Natural Number Power Function

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Definition 1 (Limit definition of derivative)**.** *Given some real valued function f, we have that the derivative of f is defined as*

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Theorem 1. *Given some $x \in \mathbb{R}$ and $a \in \mathbb{Z}$, we have that $\frac{d}{dx} x^a = ax^{a-1}$.*

Proof. We begin by taking the limit definition of the derivative and substituting in for $f(x) = x^a$. Doing so, we obtain

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^a - x^a}{\Delta x}.$$

Note now the binomial expansion

$$(x + \Delta x)^a = \sum_{k=0}^{a} \frac{a!}{(a-k)!k!} x^{a-k} \Delta x^k.$$

By taking the first few terms, we can equivalently write this as

$$\frac{a!}{a!} x^a + \frac{a!}{(a-1)!} x^{a-1} \Delta x + \sum_{k=2}^{a} \frac{a!}{(a-k)!k!} x^{a-k} \Delta x^k.$$

Simplifying the first and second terms yields

$$x^a + ax^{a-1} \Delta x + \sum_{k=2}^{a} \frac{a!}{(a-k)!k!} x^{a-k} \Delta x^k.$$

We can easily see that by factoring out an Δx^2 term and altering the starting k value to $k = 2$, we can rewrite the summation as

$$\sum_{k=2}^{a} \frac{a!}{(a-k)!k!} x^{a-k} \Delta x^k = \Delta x^2 \sum_{k=2}^{a} \frac{a!}{(a-k)!k!} x^{a-k} \Delta x^{k-2}.$$

Thus, this full expression can be written simply as

$$(x + \Delta x)^a = x^a + ax^{a-1} \Delta x + \Delta x^2 \sum_{k=2}^{a} \frac{a!}{(a-k)!k!} x^{a-k} \Delta x^{k-2}.$$

Returning to our original problem, then, we can rewrite our limit as

$$\lim_{\Delta x \to 0} \left[\frac{(x^a + ax^{a-1} \Delta x + \Delta x^2 \sum_{k=2}^{a} \frac{a!}{(a-k)!k!} x^{a-k} \Delta x^{k-2}) - x^a}{\Delta x} \right].$$

Notice by addition that the x^a and $-x^a$ terms cancel. This gives the expression

$$\lim_{\Delta x \to 0} \left[\frac{ax^{a-1} \Delta x + \Delta x^2 \sum_{k=2}^{a} \frac{a!}{(a-k)!k!} x^{a-k} \Delta x^{k-2}}{\Delta x} \right].$$

From the numerator and denominator cancel by division Δx terms, leaving only

$$\lim_{\Delta x \to 0} \left[ax^{a-1} + \Delta x \sum_{k=2}^{a} \frac{a!}{(a-k)!k!} x^{a-k} \Delta x^{k-2} \right].$$

Allowing Δx to tend to zero, we see that this expression simplifies down at last to ax^{a-1}, concluding the proof. $\square$

Corollary 1. *Given $c \in \mathbb{R}$ and $a \in \mathbb{Z}$, we have for any real valued power function f,*

$$\frac{d}{dx}[cf(x)] = c[\frac{d}{dx} f(x)].$$

Proof. This is simple to see, but I'll provide a proof anyways. First, let's take the limit definition of the derivative, which states that given some real valued function f, the derivative of f is defined as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Now, given some function $f(x) = x^a$, we should have that $(cf(x))' = cf'(x)$. First, let $g(x) = cf(x)$, which is to say $g(x) = cx^a$, and now show that $g'(x) = cf'(x)$. So, by Theorem 1, we get that our righthand side is equal to cax^{a-1}. Take note of the fact that this is equivalent to $cf'(x)$.

By comparison, the lefthand side can be expressed as

$$g'(x) = \lim_{\Delta x \to 0} [\frac{c(x + \Delta x)^a - cx^a}{\Delta x}].$$

We can factor out a c here to get

$$\lim_{\Delta x \to 0} c[\frac{(x + \Delta x)^a - x^a}{\Delta x}].$$

Of course, we already have by definition that

$$\lim_{\Delta x \to 0} [\frac{(x + \Delta x)^a - x^a}{\Delta x}] = f'(x),$$

and so, we can rewrite this as $cf'(x)$, showing that the two sides are equal. Thus, we have that $\frac{d}{dx}[cf(x)] = c[\frac{d}{dx} f(x)]$, as needed. $\square$

Theorem 2 (Power Rule)**.** *Let $a, n \in \mathbb{N}$ and $x \in \mathbb{R}$. Here, we state that*

$$(\frac{d}{dx})^n (x^a) = \frac{a!}{(a-n)!} x^{a-n}$$

That is, the nth derivative can be found by use of the factorial operator.

Proof. We'll proceed by induction. First, we check the base case, which is to say when $n = 0$. Given this, we have that $(\frac{d}{dx})^0(x^a) = \frac{a!}{(a-0)!}x^{a-0}$. This clearly simplifies down to $x^a = \frac{a!}{a!}x^a$, and so finally, $x^a = x^a$. This proves the base case, meaning we can now continue by assuming that the nth case works and therefore showing that the $n+1$ case must work as well.

Here requires a bit of a trick, which is to see that $(\frac{d}{dx})^{n+1}[x^a] = (\frac{d}{dx})^n[\frac{d}{dx}x^a]$. That is, taking the derivative of a power function $n+1$ times involves taking the derivative n times and then once more. So, we can take the inductive hypothesis and differentiate on both sides. This gives us the statement

$$\frac{d}{dx}\left[(\frac{d}{dx})^n[x^a]\right] = \frac{d}{dx}[\frac{a!}{(a-n)!}x^{a-n}].$$

There are a few places we can simplify this expression. First, see that because of the discussion on the $n+1$ derivative above, we can rewrite our equation as

$$(\frac{d}{dx})^{n+1}(x^a) = \frac{d}{dx}[\frac{a!}{(a-n)!}x^{a-n}].$$

Of course, since we have that $a, n \in \mathbb{N}$, we know $\frac{a!}{(a-n)!}$ will be equivalent to some real number c. Likewise, $a - n = d$ for some integer d. Thus, we can rewrite the expression above simply as

$$(\frac{d}{dx})^{n+1}(x^a) = \frac{d}{dx}[cx^d].$$

Interesting! Note that we can apply Corollary 1 here to obtain

$$(\frac{d}{dx})^{n+1}(x^a) = c\frac{d}{dx}[x^d].$$

Now, applying Theorem 1, we get

$$(\frac{d}{dx})^{n+1}(x^a) = cdx^{d-1}.$$

From here we would like to resubstitute for c and d. Doing so grants us the expression

$$(\frac{d}{dx})^{n+1}(x^a) = \frac{a!}{(a-n)!}(a-n)x^{a-n-1}.$$

The rest of the proof is only a case of simplification. Bring that $(a-n)$ into the fraction and get

$$(\frac{d}{dx})^{n+1}(x^a) = \frac{a!(a-n)}{(a-n)!}x^{a-n-1}.$$

Now, note that $\frac{(a-n)}{(a-n)!} = \frac{1}{(a-n-1)!}$. Thus, we obtain

$$(\frac{d}{dx})^{n+1}(x^a) = \frac{a!}{(a-n-1)!}x^{a-n-1}.$$

Finally, see that $a - n - 1 = a - (n+1)$. This allows us to rewrite the expression as

$$(\frac{d}{dx})^{n+1}(x^a) = \frac{a!}{(a-(n+1))!}x^{a-(n+1)}.$$

By the inductive hypothesis, this concludes the proof. □